Gravitational acceleration on the Earth's surface

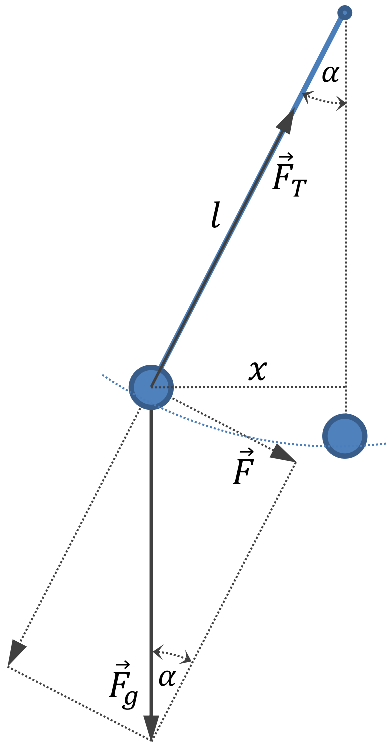
Introduction

The aim of the exercise is to determine the gravitational acceleration on the Earth's surface using a mathematical pendulum and to examine the damped physical pendulum motion, for which the so-called logarithmic decrement of damping is determined.

The body moves with acceleration (g) during a free fall at the Earth's surface when the air resistance is negligible. Due to the non-spherical shape of the Earth and its rotary motion, the acceleration changes from the value 9.764 m/s^2 at the equator, to the value of 9.863 m/s^2 at the pole. For example, for 45° of latitude, the gravitational acceleration is 9,806 m/s^2. Since the acceleration during free fall, according to the law of universal gravity, depends on the distance of the body from the center of the Earth, for the mentioned latitude, at the height of 16 km, it is 9.757 m/s^2.

Determining the value of the gravitational acceleration directly from the measurements of free fall of the bodies does not give results with the desired accuracy. The most accurate measurement methods are based on observing the orbits of artificial satellites. The value of g can, however, be determined with satisfactory accuracy by means of a simple pendulum.

A pendulum is a weight suspended from a pivot so that it can swing freely. When a pendulum is displaced sideways from its equilibrium position, a restoring force due to gravity will accelerate it back toward the equilibrium position. When released, the restoring force acting on the pendulum's mass causes it to swing back and forth. The simple gravity pendulum is an idealized mathematical model of a pendulum. This is a weight on the end of a massless cord suspended from a pivot, without friction. When given an initial push, it will swing back and forth at a constant amplitude. Real pendulums are subject to friction and air resistance, so the amplitude of their swings declines.



The pendulum of length l shown in Fig. 1.1 is pivoted by an angle α. The gravitational force (F_g) and tension force (F_T) act on the ball of mass m. The resultant of both these forces (F), determining the pendulum's motion, is the tangent component of the force of gravity.

$$F = F_g sin(\alpha) = mg sin(\alpha). \tag{1.1}$$

In a right-angled triangle containing sides with lengths l and x

$$sin(\alpha) = \frac{x}{l}. \tag{1.2}$$

Inserting 1.2 into 1.1 we have:

$$F = mg\frac{x}{l}. \tag{1.3}$$

For small angles (α) the force F is approximately equal to its horizontal component (F_x) (not shown in Fig. 1.1), which is proportional to the displacement (x) of the pendulum from the equilibrium position, according to the formula:

$$F \cong F_x = kx, \tag{1.4}$$

Fig 1.1 Distribution of forces acting on a ball in a simple pendulum.

where k is force constant. Substituting 1.4 to 1.3 we get

$$kx = mg\frac{x}{l} \tag{1.5}$$

and after deletion x

$$k = \frac{mg}{l}. \tag{1.6}$$

k can be expressed also as a function of free oscillation pulsation (ω),

$$k = m\omega^2, \tag{1.7}$$

which in turn is dependent on the vibration period (T) as follows:

$$\omega = \frac{2\pi}{T}. \tag{1.8}$$

Taking into account 1.8 and 1.7, the formula 1.6 can be expressed as follows:

$$m\left(\frac{2\pi}{T}\right)^2 = \frac{mg}{l}. \tag{1.9}$$

After simple transformations and deletion of m we get the formula for Earth's acceleration:

$$g = 4\pi^2\frac{l}{T^2}, \tag{1.10}$$

which can be determined knowing the length of the pendulum (l) and the period of vibration (T).

Any swinging rigid body freely rotating about a fixed horizontal axis is called a physical pendulum. The appropriate equivalent length (L) for calculating the period of any such pendulum is the distance from the pivot to the center of oscillation. This point is located under the center of mass at a distance from the pivot traditionally called the radius of oscillation, which depends on the mass distribution of the pendulum. If most of the mass is concentrated in a relatively small size compared to the pendulum length, the center of oscillation is close to the center of mass. The radius of oscillation or equivalent length (L) of any physical pendulum can be shown to be

$$L = \frac{I}{mR} \tag{1.11}$$

where I is the moment of inertia of the pendulum about the pivot point, m is the mass of the pendulum, and R is the distance between the pivot point and the center of mass. Substituting this expression in 1.10, the period of a physical pendulum is given by

$$T = 2\pi\sqrt{\frac{I}{mgR}}. \qquad 1.12)$$

If the movement of the pendulum takes place in a material medium, then due to the resistance of the medium, which we treat as a damping force, the vibrations will disappear. Regardless of the nature of the medium, the damping force F_D, if the speed (v) is small, can be represented by the formula:

$$F_D = -b \cdot v = -b\frac{dx}{dt}. \tag{1.13}$$

The constant factor b is called the damping constant, and the minus sign in the above formula is a consequence of the fact that the force F_D is always directed against the direction of motion. Taking into account the effect of force (1.13), we can write the equation of motion for damped oscillations, according to the second principle of dynamics:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0. \tag{1.13}$$

It is a differential equation of damped vibrations, the solution of which is a function

$$x(t) = A_0 e^{-\beta t} sin(\omega_1 t + \varphi), \tag{1.14}$$

where $\beta = \frac{b}{2m}$ – damping factor, $\omega_1 = \sqrt{\omega^2 - \beta^2}$ – pulsation of damped vibrations. From formula (1.14) we see that due to the damping force

- the amplitude of vibrations decreases with time according to the dependence

$$A(t) = A_0 e^{-\beta t}, \tag{1.15}$$

- vibration pulsation is less than for free vibrations

$$\omega_1 = \sqrt{\omega^2 - \beta^2} \leq \omega \tag{1.16}$$

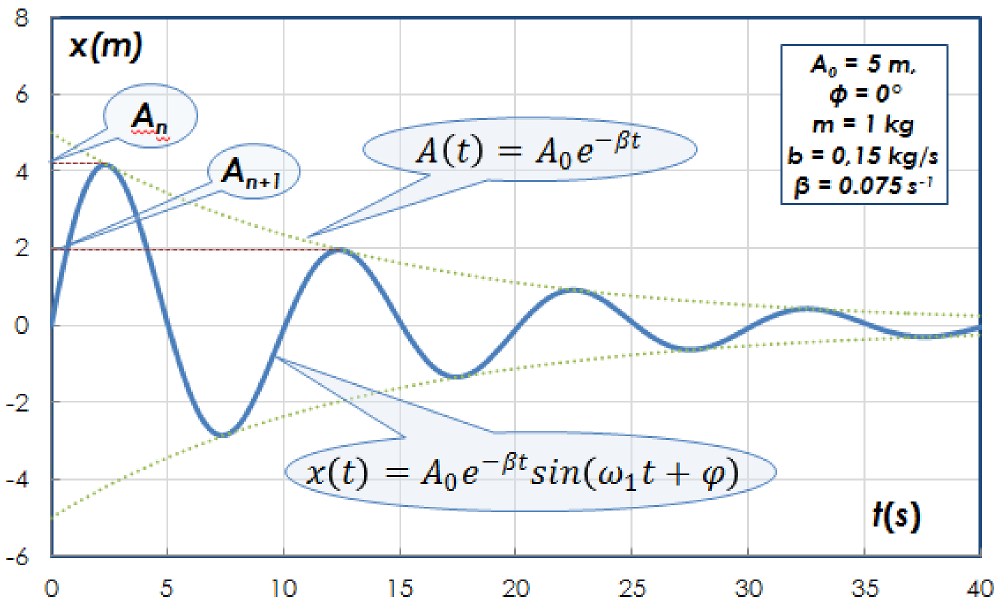


Fig. 1.2 An example of the dependence of the point's displacement from the equilibrium position in the damped motion. The assumed values of the initial amplitude (A_0), initial phase (φ), mass of the vibrating body (m), damping constant (b) and damping factor (β) are included in the figure.

Another physical quantity characterizing damped vibrations is so-called "logarithmic damping decrement" (D). This is the natural logarithm of the ratio of any two successive amplitudes A_n and A_{n+1} at times t and $t+T$.

$$D = ln\frac{A_n}{A_{n+1}} = \frac{A_0 e^{-\beta t}}{A_0 e^{-\beta(t+T)}} = ln\left(e^{\beta T}\right) = \beta T. \tag{1.17}$$

Due to the ease of determination of D and T, the formula (1.17) can be used to calculate the damping factor $\beta = \frac{D}{T}$ (1.18) and damping constant $b = 2\beta m$ (1.19).

Description of the laboratory set

The pendulum set consists of four balls (two wooden and two metal) suspended on threads of various lengths. The whole was placed on a common tripod as illustrated in Fig. 1.3.

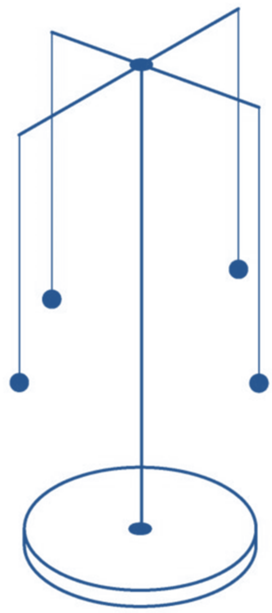


Fig 1.3 Simple pendulum set

The physical pendulum used in the exercise was made of a set of plates in a frame suspended on two long cords (about 4 m), as shown in Figure 1.4. Single plates can be positioned transversely or longitudinally to the direction of movement.

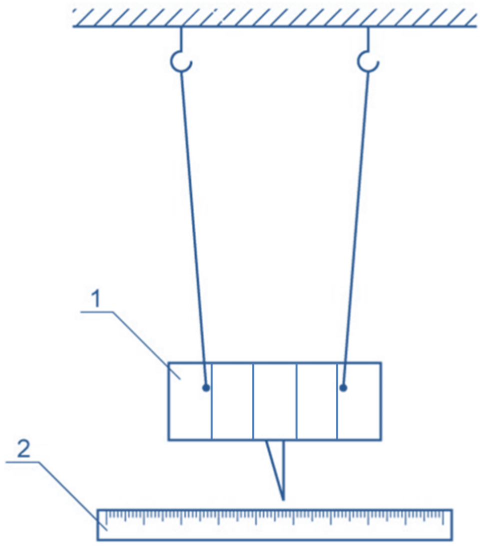


Fig. 1.4 Physical pendulum; 1- the set of plates suspended on two long cords, 2- the scale of the ruler.

Proceeding

Gravitational acceleration

1. Determine the length of the pendulum by specifying the length of the thread l_t and the radius of the ball. The length is the sum of these two quantities. When measuring the length of the thread, i.e. the distance between the ball and the point of suspension of the thread, use a cathetometer - the instrument for measuring vertical distances in cases where a scale cannot be placed very close to the points whose distance apart is desired. Determine the radius of the ball, by measuring the diameter of the ball with a caliper. It guarantees good accuracy. Of course, the radius is obtained by dividing the diameter by two. Repeat the measurements three times.
2. Incline the ball on an angle not exceeding 5º and then let it go freely, thus introducing the pendulum into swaying. After a few full cycles of swings, start the stopwatch at the moment of maximum displacement from balance position and measure the time of 30 full swing cycles.

3. Perform the measurements described in points 1 and 2 for each of the pendulum, and save the results in the table.

Table of measurements and calculation results

Type of ball	Thread length l_t (m)	Diameter of the ball d (m)	Length of the pendulum $l_{av}=(l_t+d_{av}/2)$ (m)	Duration of 30 periods (s)	Period T (s)	Earth's acceleration $g = 4\pi^2 l_{av}/T_{av}^2$ (m/s^2)
Aluminium	**0.400**•	**0.0248**	**0.4124**	**38.60**	**1.287**	**9.829**
Wooden	**0.295**	**0.0292**				
Brass	**0.660**	**0.0247**				
Steel	**0.820**	**0.0398**				

- The data are given for an example. Students should write the results of their own measurements.
- The data marked in blue must be saved during classes, others may be completed later.

Determination of logarithmic damping decrement.

1. Determine with the teacher the geometry of the plates in the physical pendulum.
2. Incline the physical pendulum from the balance position beyond the scale and release it freely.
2. If the pendulum movement is free from transverse vibrations, read from the scale 10 successive amplitudes starting with the first readable one. Read the amplitudes on one side in relation to the equilibrium position.
3. Enter the results of measurements into the table

Table of measurements

n	1	2	3	4	5	6	7	8	9	10
A_n (cm)	**48.0**	**39.5**	**36.0**	**32.5**	**29,0**	**26.5**	**24.0**	**21.0**	**19.0**	**17.5**

- The data marked in blue are given for an example.

4. Measure the time of full 10 swing cycles of the physical pendulum.
5. The mass of the pendulum stored on the plate (m = 3,851 kg, $u(m)$ = 0,010 kg).

Processing of the data

Earth's acceleration

1. Using the formula (1.10) determine the value of g for all pendulums.
2. Estimate the standard uncertainties u(T) and u(l) using type B of evaluation or take them as: 0.20 s and 0.0020 m, respectively.
3. Calculate the combined uncertainty $u_c(g)$ using following formula:

$$u_c(g) = \sqrt{\left[\frac{\partial g}{\partial l} \cdot u(l)\right]^2 + \left[\frac{\partial g}{\partial T} \cdot u(T)\right]^2} = \sqrt{\left[\frac{4\pi^2}{T^2} \cdot u(l)\right]^2 + \left[\frac{-2T \cdot 4\pi^2 l}{T^4} \cdot u(T)\right]^2}, \qquad (1.20)$$

4. Set of final results of calculations with appropriate number of significant digits.
5. Using the concept of expanded uncertainty, compare the received acceleration g with a reference value. What conclusions result from this analysis?

Logarithmic damping decrement

1. Using the relationship 1.17,

$$D = ln\frac{A_n}{A_{n+1}}, \qquad (1.21)$$

calculate the values of the logarithmic damping decrement for each pair of successive amplitudes. Then calculate the average value of the decrement.

2. Estimate the standard uncertainty $u(\bar{D})$ using type A of evaluation

$$u(\bar{D}) = \sqrt{\frac{\sum_{i=1}^{n}(\bar{D}-D_i)^2}{n(n-1)}}. \qquad (1.22)$$

3. Determine the damping factor (β), damping constant (b) and their uncertainties on the base of the following formulas:

$$\beta = \frac{\bar{D}}{T},$$
$$u(\beta) = \sqrt{\left(\frac{u(\bar{D})}{T}\right)^2 + \left(\frac{-\bar{D} \cdot u(T)}{T^2}\right)^2}, \qquad (1.23)$$

$$b = 2\beta m,$$
$$u(b) = \sqrt{\left(2m \cdot u(\beta)\right)^2 + \left(2\beta \cdot u(m)\right)^2}. \qquad (1.24)$$

Supplementary literature

1. Andrzej Kubiaczyk, Evaluation of Uncertainty in Measurements, Warsaw University of Technology, http://www.if.pw.edu.pl … .